Dynamic Time Step for One-Dimensional Overland Flow Kinematic Wave Solution

Fouad H. Jaber[1] and Rabi H. Mohtar[2]

Abstract: Kinematic wave theory is widely used in modeling a variety of hydrologic processes. Results of applying the kinematic wave overland flow solution using different time steps showed that the conventionally used stability criterion known as the Courant condition fails to give a time step estimate that ensures stable and accurate numerical solutions. Accordingly, a new accuracy-based dynamic time step estimate for the one dimensional overland flow kinematic wave solution is developed. The newly developed dynamic time step estimates are functions of the mesh size, watershed slope, roughness, excess rainfall, and time of concentration. The new criteria were developed by comparing the consistent formulation of the Galerkin-Crank Nicholson numerical solution of the kinematic wave equation to the characteristic method-based analytical solution, using different time steps and meshes. For each simulation, characterized by boundary and initial conditions and mesh size, an optimal time step that integrates the problem within 5% error was determined. The series of mesh sizes and corresponding optimal time steps were used to develop the dynamic time step. The time step criteria were tested on a variety of problems, including a steady state and time varying rainfall scenarios, and proved to be adequate for accurate and stable results within an efficient computational time. The criteria can be easily integrated in flow routing models to select the optimal time step with minimal user input.

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Introduction

The kinematic wave theory has been researched and reported extensively in hydrology literature since Lighthill and Whitham introduced it (Lighthill and Whitham 1955). It has been applied in many areas and is now well established for modeling a variety of hydrologic processes. Growing environmental and ecological concerns have increased the role of the kinematic wave theory in describing and modeling environmental and hydrologic processes (Singh 1996). The kinematic wave equations, resulting from simplification of the Saint-Venant equations, have many advantages. Among these are the possibility of analytical solutions for simple geometries and fewer boundary conditions as compared with the far more complex Saint-Venant equations. Hjelmfelt (1981), Parlange et al. (1981), Govindaraju et al. (1988, 1990), and Singh (1996) have provided good insights for generating analytical or semianalytical solutions for the kinematic wave approximation. However, except for Singh (1996), who provided solutions for problems with rainfall, slope, and roughness varying in space, their analyses dealt only with constant or time-varying rainfall. Problems with spatial variation of rainfall or surface characteristics, such as roughness and slope, are still not well documented in the literature. Problems that can be solved analytically require that rainfall, slope, and roughness vary in space, according to a prescribed mathematical relation. Thus, problems in which rainfall and surface characteristics vary randomly in space have no analytical solution.

Higher computational power and the development of spatial data analysis tools such as geographic information systems (GIS) have made the use of numerical methods to solve these problems much easier. The one-dimensional kinematic wave equation is governed by the continuity equation

$$V\frac{\partial h}{\partial x}+h\frac{\partial V}{\partial x}+\frac{\partial h}{\partial t}=r(x,t)-i(x,t) \tag{1}$$

and the conservation of momentum equation, which is reduced to

$$S_0=S_f \tag{2}$$

by the kinematic wave assumption, where V = depth averaged velocity; h = vertical flow depth; $r(x,t)$ and $i(x,t)$ = rainfall and infiltration rates, respectively; S_0 = element bottom slope; S_f = friction slope; and x and t represent the space and time domains, respectively. The derivation of Eqs. (1) and (2) is given by Chow et al. (1988) and Henderson (1966), among many others.

The flow rate per unit area equivalent to the value of V times h and depth in Eq. (1) can be related by a parametric friction loss equation in which S_0 is substituted for S_f as follows:

$$q=\alpha_x h^{\beta} \quad \text{or} \quad V=\alpha_x h^{\beta-1} \tag{3}$$

where, using Manning's equation, $\alpha_x=(S_{ox}^{1/2}/n)$ and $\beta=(5/3)$, in which n = Manning roughness coefficient and V = flow velocity, assuming SI units.

Eq. (1) is the differential form of continuity. Together with the energy loss relation (Manning, Chezy, or Darcy-Weisbach) shown in Eq. (3), these form the governing kinematic wave equations.

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The validity of the kinematic simplifying assumption is considered accurate to within 10% (Woolhiser and Liggett 1967), if

$$\frac{LS_0}{\mathsf{F}^2 h_0} > 10 \tag{4}$$

where L = length of domain; F = Froude number and is equal to $\mathsf{F} = (u_0/(gh_0)^{0.5})$; and h_0 and u_0 = depth and velocity of flow at the downstream end of the overland flow plane under steady-state conditions, respectively.

Integrating Eq. (1) with the boundary and initial conditions over space and using the consistent formulation of the finite-element method yields the time dependent system of ordinary differential equations (ODEs):

$$[C]\{h\}_{\text{new}} = [C]\{h\}_{\text{old}} - \Delta t[B]((1-\theta)\{q\}_{\text{old}} + \theta\{q\}_{\text{new}}) + \Delta t((1-\theta)\{F\}_{\text{old}} + \theta\{F\}_{\text{new}}) \tag{5}$$

where $[C]$ and $[B]$ are matrices resulting from the finite-element solution in space and are referred to in finite-element analysis as capacitance and gradient matrices; F = forcing term, i.e., the lateral inflow that is the rainfall excess for an overland flow plane; h = depth of flow; q = flow per unit area equivalent to the value of V times h; θ is a factor that determines the type of finite difference scheme solved; and old and new refer to the previous time step and the actual time step. $\theta = 0$, 0.5, 0.67, and 1 for Euler, central difference, Galerkin, and backward difference schemes, respectively. Mohtar and Segerlind (1998) showed that the central difference scheme ($\theta = 0.5$) is the most accurate single step scheme among the four schemes; thus, it will be used in the current study.

Eq. (5) is coupled with Eq. (3) to generate a nonlinear system of equations that will be solved for the flow depth at each node and time step. For a more detailed description of the finite-element formulation and finite difference solution in time, see Vieux et al. (1990a, b).

Numerical methods require discretization in both space and time. The size of the element and the time step are crucial to the stability and the accuracy of the numerical scheme used. Previous studies (Courant et al. 1956; Vieux and Segerlind 1989) have shown that the actual time step used in the time integration scheme must not be longer than the time during which a gravity wave front may propagate through the system, or longer than the time step variation in the forcing function. The prior condition is known as the Courant condition (Courant et al. 1956). The Courant time step for each element may be computed using the celerity of a gravity wave by

$$T_{\text{Courant}} < \frac{\Delta x}{(gh)^{1/2}} \tag{6a}$$

where T_{Courant} = Courant time step in seconds; Δx = distance increment or element length in meters; g = gravitational acceleration in m/s^2; and h = downstream equilibrium flow depth in meters.

For the kinematic wave assumption, the critical time step T_{Courant} can be expressed as

$$T_{\text{Courant}} = \frac{lS_0^{0.15}c^{0.3}}{g^{0.5}L^{0.3}r_e^{0.3}n^{0.3}} \tag{6b}$$

where l = element length in meters; S_0 = watershed slope in m/m; c is a constant equal to 1.0 in the SI system and 1.49 in English units; g = gravitational acceleration in m/s^2; L = slope length in meters; r_e = rainfall excess rate in m/s; and n = Manning's roughness factor. Numerical errors arise if the time step exceeds the condition in Eq. (6*a*). This condition is limited by its inability to deal with cases where slope, roughness, or rainfall excess input varies from element to element. This condition, however, is a stability criterion and not an accuracy criterion. Lyn and Goodwin (1987) showed that, although a given difference scheme may be stable, its convergence may be poor. In other words, if a time step has low convergence, then it is too large for accuracy (Mohtar and Segerlind 1995). Limited research has been conducted on criteria for a time step that would ensure stability as well as accuracy of the kinematic wave solution. The task of choosing the proper time step has often been considered a matter of experience. Bajracharya and Barry (1997) have stated that reducing the time step does not necessarily result in more accurate solutions. They added that optimal solutions are yet to be found. Bajracharya and Barry (1997) have generated a spatial step based on the truncation error in the finite difference solution of the Muskingum-Cunge form of the linear kinematic wave problem. The time step, then, is generated for the calculated Δx using the Courant condition. Thus, to get optimum results one must vary the Courant condition to get a Δt from the calculated Δx that would ensure the accuracy of the problem. This still requires some trial and error, while ignoring the temporal error in the kinematic wave problem. In addition, the Muskingum-Cunge model is limited to situations of flood forecasts where simplicity and rapidity of computations are sought rather than accuracy (Cunge 1999). Hromadka and DeVries (1988) argued after a series of tests varying Δx and Δt that the use of the kinematic wave method for channel routing needs evaluation for use in hydrologic models unless guidelines are developed to control the arbitrary use of the kinematic wave in design studies. They added that kinematic wave programs need internal checks to select Δx and Δt such that an accurate solution is achieved. This clearly shows the need for time step criteria, which can be used in overland flow models to ensure accurate solutions for the kinematic wave equation. The criteria should be part of a user input-free hydrologic modeling environment, thus eliminating the trial-and-error procedures that usually accompany the time step selection in watershed modeling.

The objective of this study is to develop and evaluate an accuracy-based dynamic time step estimate for the numerical solution of the one-dimensional kinematic wave problem. The criteria can be part of a user input-free hydrologic modeling environment. The complexity of the mathematical problem makes an analytical approach to this problem very difficult. The approach in this study utilizes numerical experimentation.

Methodology for Developing Dynamic Time Step Criteria

Overall Methodology

The one-dimensional kinematic wave overland flow problem, governed by Eqs. (1) and (3) together with specified boundary and initial conditions, is solved using constant and varying rainfall and Manning's α for a certain mesh using different time steps. For each time step, the error between an analytical solution obtained by the characteristics method and the numerical solution was computed. A time step versus error graph was generated for this series of solutions. As the value of the time step increased, the error was expected to grow, provided that the spatial discretization error is small. A time step value exists that will integrate the problem within a specified error. Using a smaller time step might result in an unnecessarily longer computational time, while

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a larger time step might result in large errors and might even lead to instability in the solution. The specified problem was solved using several mesh sizes to generate a relation between mesh size and optimal time step. The results are summarized in a regression equation that was used to define a time step estimate. The time of concentration, which depends on the excess rainfall, the slope, and the roughness factor of the watershed, was introduced in the equation as a problem specific factor so the criteria can be used for other problems. The time step estimation equations were tested using a different problem to ensure the validity of the results.

Numerical Experiment Problem Definition

Accuracy in this analysis was assessed by comparing the numerical solution to the analytical or exact solution in a numerical experiment. These solutions are applicable for simple problems where the spatial distribution of rainfall and the spatial distribution of surface characteristics (Manning's roughness and slope) can be expressed by a mathematical relation. The experimental scenarios were chosen accordingly. Four cases were considered in this study:

1. r_e and α constant;
2. r_e varying in space and α constant;
3. r_e constant and α varying in space; and
4. r_e and α varying in space

where r_e=rainfall excess $[r(x,t)-i(x,t)$ from Eq. (1)]; and α =Manning's friction parameter of Eq. (3).

In order to illustrate the solution accuracy requirement in terms of time steps and element sizes, nodal values of water depths were computed for 5, 10, 20, 25, 50, and 75 elements for each case. Various time steps were used to integrate the system of ordinary differential Eq. (6). Each scenario was solved using the following conditions: (1) contributing area of unit width has a length L equal to 152.4 m (500 ft); (2) the average rainfall excess intensity is 2.74 cm/h (2.5E−5 ft/s); (3) the average bottom slope is 0.05; and (4) the average Manning's roughness coefficient is 0.035. For cases where r and α varied in space, the following relations were used:

$$r_e=-0.036x+5.49$$

$$\alpha=6.015\cdot e^{0.00984x}$$

where x=length coordinate along the hillslope in meters.

The Courant condition [Eq. (6)] of each simulation was calculated as well as the time of concentration according to the following equation (Singh 1996):

$$t_c=\frac{1}{\beta}\int_0^L\left[\frac{1}{\alpha(\eta)}\right]^{1/\beta}\left[\int_0^\eta r_e(\xi)d\xi\right]^{(1-\beta)/\beta}d\eta \tag{7}$$

where $\beta=5/3$; L=slope length; and η and ξ=dummy variables.

The storm duration was set to exceed the time to concentration so that preequilibrium, equilibrium, and postequilibrium conditions could be evaluated; thus, the storm length or the duration of continuous rainfall was set to be 0.4 hours, based upon element properties. The total simulation time was 1 hour.

Error Criteria

The average percentage error was computed for each numerical simulation by the following equation:

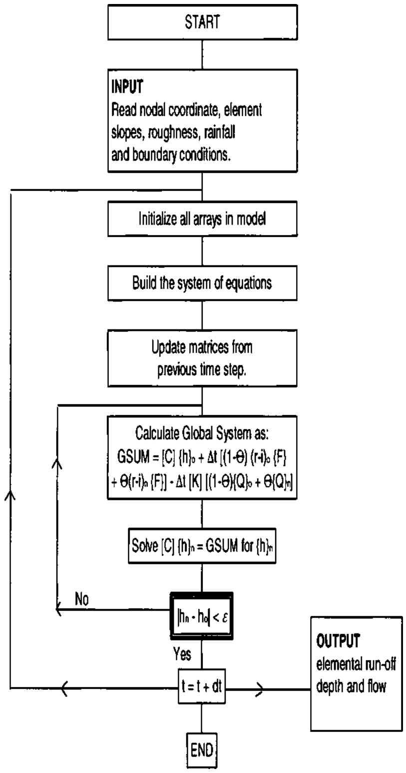

Fig. 1. 2-DSTREAM Flow Chart modified for this research

$$e=\frac{1}{n}\left[\sum_{i=1}^{n}\left(\frac{\sum_{j=1}^{m}|qa_{ij}-qn_{ij}|}{\sum_{j=1}^{m}qa_{ij}}\right)\right]\times 100 \tag{8}$$

where e=average percentage error; m=number of sampling points in space; n=number of sampling points in time; q_a =analytical solution for flow; and q_n=numerical solution for flow.

Numerical Solution

A finite-element-based overland flow model 2DSTREAM, originally developed by Vieux et al. (1990a) and modified for the purpose of this study, was used for solving the uncoupled sets of overland flow Eqs. (1) and (5). Fig. 1 shows the modified model flow chart. Watershed input data such as nodal coordinates, elemental slopes, and roughness are read at the beginning of each run. The forcing function $r_e(x)$ then is calculated. Using the element and force vector information, the system of equations is built, updated, and solved for new flow depth (h) values.

The solution for h is implicit and requires iteration until convergence to a specified tolerance value ϵ. After convergence on h, the time is incremented and the solution proceeds in the same manner by updating the time matrices and evaluating new h values. The model used has been validated for different rainfall and slope conditions (Vieux et al. 1990a; Jaber and Mohtar 1998).

Analytical Solution

Analytical solutions can be derived for linear problems with simple geometrical domains where functions for rainfall and watershed characteristics, such as slope and roughness, are predetermined (Chow et al. 1988; Singh 1996). Singh (1996) presented a general form of the analytical solution for rainfall excess and

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Manning's α, both varying in space only. Eq. (9) represents the rising and constant parts of the hydrograph, while Eq. (10) is for the receding limb of the hydrograph:

$$h(x)=\left[\frac{1}{\alpha(x)}\int_0^x r_e(\xi)d\xi\right]^{1/\beta} \tag{9a}$$

$$t(x)=\frac{1}{\beta}\int_0^x\left[\frac{1}{\alpha(\eta)}\right]^{1/\beta}\left[\int_0^\eta r_e(\xi)d\xi\right]^{(1-\beta)/\beta}d\eta \quad \text{for} \quad 0\leqslant t\leqslant t_r \tag{9b}$$

and

$$h(x,x^*)=\left[\frac{1}{\alpha(x)}\int_0^{x^*} r_e(\xi)d\xi\right]^{1/\beta} \tag{10a}$$

$$t(x,x^*)=tr+\frac{1}{\beta}\left[\int_0^{x^*} r_e(\xi)d\xi\right]^{(1-\beta)/\beta}\int_{x^*}^x\left[\frac{1}{\alpha(\eta)}\right]^{1/\beta}d\eta \quad \text{for} \quad t_r\leqslant t\leqslant t_f \tag{10b}$$

where r_e = input rainfall excess rate; $\beta=5/3$; x = space coordinate; α is a factor of roughness and slope determined from Manning's equation; t_r = storm duration in hours; t_f = total simulation time; x^* = intersection of the characteristic curve passing through the origin with the $t=t_r$ line; and ξ and η = dummy variables. Using Eq. (9) for a fixed value of x, a hydrograph may be developed to calculate depth and discharge of water as a function of time. Similarly, the water profile of the slope for a given time can also be calculated from Eq. (9).

From Eq. (10*b*), an expression for x^* can be generated as a function of t and x and used in Eq. (10*a*) to solve for the receding part of the hydrograph. This part requires an iterative solution because of the nonlinearity of the resulting equation.

For all analytical solutions that were generated [q_a in Eq. (8)], space was sampled at each node in the system, while time was sampled at 0.15, 0.30, 0.40, 0.50, 0.75, and 1.0 hour. These values were compared with the generated numerical solution q_n and used in the error term of Eq. (8).

In cases 2, 3, and 4, for points beyond the location of the wave maximum flow in the rising limb, the solution [Eq. (9)] is not valid, and thus not included in the error calculation.

Time Steps Selected

Each of the previously mentioned cases were run using fixed time steps of 0.1, 0.25, 0.50, 0.75, 1.0, and 1.2 times respective Courant time step of each simulation. This allows examination of the effect of the time step on the accuracy of the solution and assessment of the validity of the Courant condition. The numerical solution was sampled at each of the nodes and selected times and compared with the analytical solution, as previously indicated.

Dynamic Time Step Determination Process

Time step criteria for the numerical solution of the kinematic wave equation were determined following the methodology of Mohtar and Segerlind (1998, 1999a, b). To estimate the time step needed to integrate the ODE in Eq. (6), a numerical experiment was conducted with the previously mentioned conditions. The four cases involving the spatial variability of r and α were analyzed using six mesh sizes. Each of these 24 scenarios was simulated using six time steps that were selected as multiples of the Courant time step of each scenario. Analytical and numerical solutions were compared for each simulation according to Eq. (8). A response line of average error versus time step was plotted. The time step that provided 5% average error was graphically selected for each scenario. The selected time steps were plotted against the mesh size and regressed to generate an equation that would provide the time step that would integrate the kinematic wave equation with 95% average accuracy. The time of concentration, calculated using Eq. (7), was integrated into the regression as a problem specific factor. For each case, a time step dependent on the number of elements and the time of concentration was generated.

Time of Concentration Estimates

The time of concentration (t_c) is defined mathematically as the intersection of the characteristic curve $t(x,0)$ passing through the origin with the line $x=L$. The time of concentration can be defined as the time when the maximum flow is reached in the hydrograph (thus, it is also sometimes known as time to equilibrium). It is typically determined using a version of the kinematic wave equation (McCuen and Spiess 1995). Singh (1996) determined the general form of t_c given in Eq. (7).

The time of concentration for each of the three simplified cases, then would be as follows.

Case 1: α and r_e constant:

$$t_c=r_e^{(1-\beta)/\beta}\left(\frac{L}{\alpha}\right)^{1/\beta} \tag{11a}$$

Case 2: α constant and r_e varies in space:

$$t_c=\frac{1}{\beta}\left(\frac{1}{\alpha}\right)^{1/\beta}\int_0^L\left[\int_0^\eta r_e(\xi)d\xi\right]^{(1-\beta)/\beta}d\eta \tag{11b}$$

Case 3: α varies in space, r_e constant:

$$t_c=\frac{r_e^{(1-\beta)/\beta}}{\beta}\int_0^L\left[\frac{1}{\alpha(\xi)}\right]^{1/\beta}\xi^{(1-\beta)/\beta}d\xi \tag{11c}$$

Several empirical estimations can also be found in the literature (McCuen and Spiess 1995). These are only applicable for case 1 problems.

One such relation commonly used is that of Ragan and Duru (1972):

$$t_c=\frac{0.0803(nL)^{0.6}}{r_eS^{0.3}} \tag{12}$$

where t_c is in hours; n = Manning's roughness coefficient; L = length in meters; r_e = rainfall excess in cm/h; and S = slope in m/m. The analytical, numerical, and empirical times of concentration (t_c) are shown in Table 1. The numerical t_c was determined as the time when all the hillslope is contributing to the flow.

Adequacy of Courant Condition

The time steps used in developing the new criteria were fractions and multiples of the Courant time step, varying from $0.1\cdot T_{\text{Courant}}$ (Courant condition time step) to $1.2\cdot T_{\text{Courant}}$. Example plots for a 20 element mesh for case 1 (r_e and α constant in space) at a time step equal to 0.75 T_{Courant} and T_{Courant} are shown in Figs. 2 and 3. Fig. 3 shows the numerical instability for time steps equal to the Courant condition. This instability is caused by the time step being too large, which proves the inad-

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Table 1. Analytical, Numerical, and Empirical Times of Concentration in Hours

Estimate	Case 1	Case 2	Case 3	Case 4
Analytical	0.21	0.17	0.20	0.17
Numerical	0.21	0.18	0.20	0.17
Regan and Duru (1972)	0.20	N/A	N/A	N/A

equacy of the Courant time step for integrating the overland flow problem. Fig. 4 is a plot of the error against the dimensionless time step calculated as $\Delta t/T_{\mathrm{Courant}}$ for case 1. The runs for a time step equal to $1.2\cdot T_{\mathrm{Courant}}$ converged only for a mesh of five elements. The time step equal to T_{Courant} did not converge for the 25 element mesh, and time steps greater than $0.5\cdot T_{\mathrm{Courant}}$ did not converge for 50 and 75 element meshes. There is a threshold beyond which the error suddenly increases. Fig. 4 also shows that the threshold, after which the error drastically increased, varied with the mesh size. For a mesh of five elements, it was greater than $1.2\cdot T_{\mathrm{Courant}}$, while for 50 elements, it was between $0.5\cdot T_{\mathrm{Courant}}$ and $0.75\cdot T_{\mathrm{Courant}}$. This proves that the Courant condition time step is not a reliable criterion for accuracy and stability. New criteria need to be developed that would take the system characteristics into consideration.

Dynamic Time Step Estimates

The results generated in the numerical experiments correspond to error estimates for each of the six meshes and the six different time steps for each of the four cases. The error was calculated for each of the solutions that converged. The error was plotted versus the time step. The results are shown in Figs. 5–8 for cases 1–4, respectively.

As expected, error decreased considerably as the number of elements increased. It also was observed that the error remained more or less constant or increased very slightly to a certain threshold, where it increases drastically or even where no convergence occurs in the solution.

The threshold time steps were determined graphically using Figs. 5–8. The largest time step that resulted in less than 5% error was chosen for each of the six meshes selected. These time steps were divided by the time of concentration and regressed against the number of elements in each mesh. A power best fit was generated for each case line (Figs. 9–12). The power line was chosen because it best fit the shape of the generated curves.

The dynamic time step estimates, Δt in hours, for the four cases are, respectively

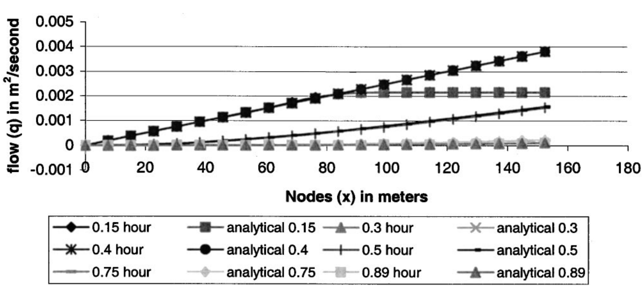

Fig. 2. 2-DSTREAM solutions compared with analytical solutions for 20 element mesh and time step of 0.75 Courant time step at different times of runoff hydrograph for Case 1 (α and r_e constant)

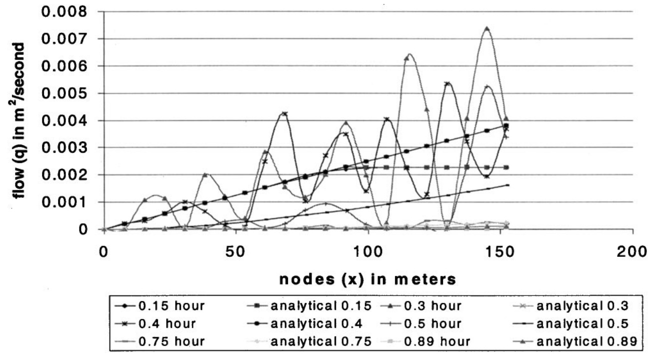

Fig. 3. 2-DSTREAM solutions compared with analytical solutions for 20 element mesh and time step equivalent to Courant time step at different times of runoff hydrograph for Case 1 (α and r_e constant)

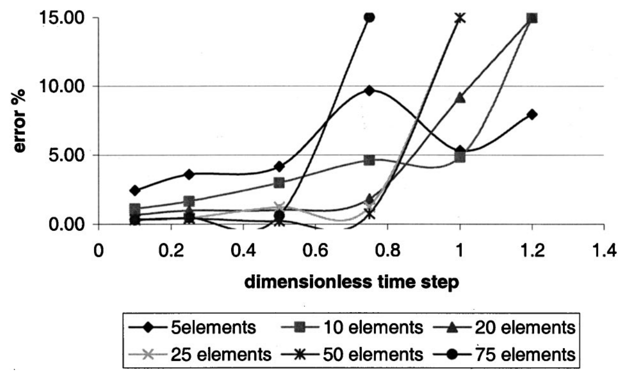

Fig. 4. Error versus time step as fraction of Courant time step for Case 1 (α and r_e constant)

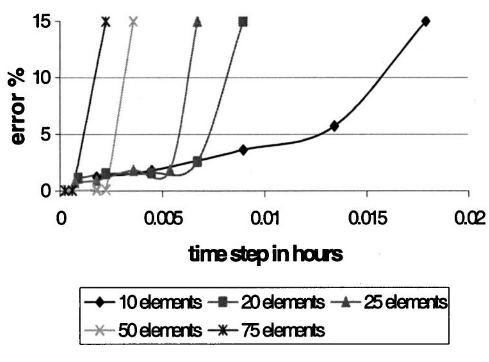

Fig. 5. Error versus time step in hours for Case 1

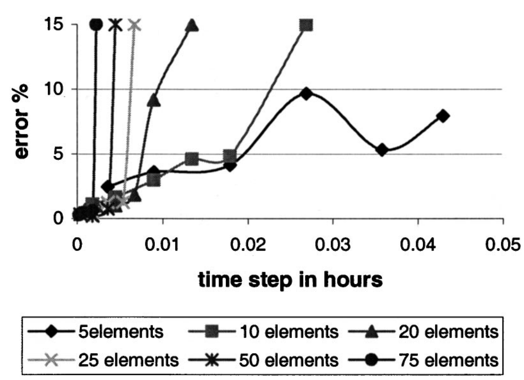

Fig. 6. Error versus time step in hours for Case 2

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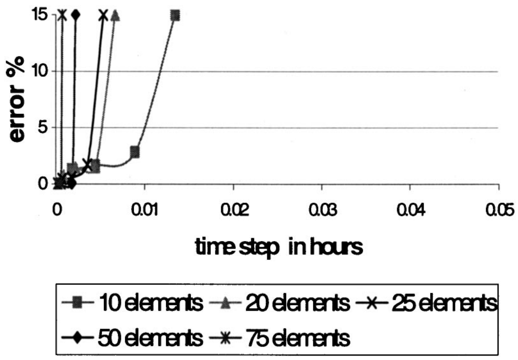

Fig. 7. Error versus time step in hours for Case 3

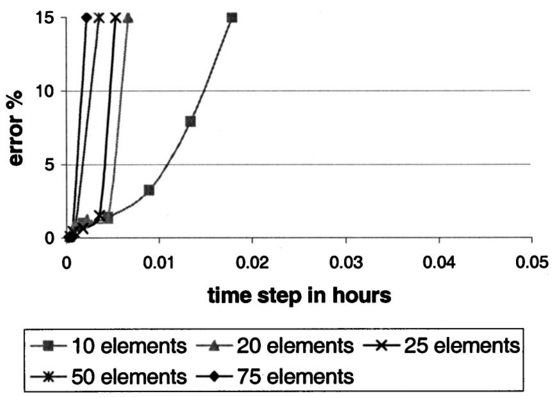

Fig. 8. Error versus time step in hours for Case 4

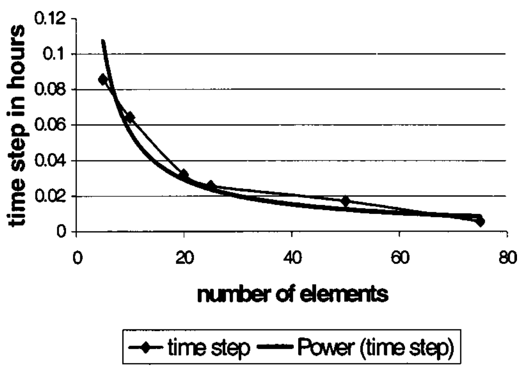

Fig. 9. Regression of time step versus number of elements for Case 1

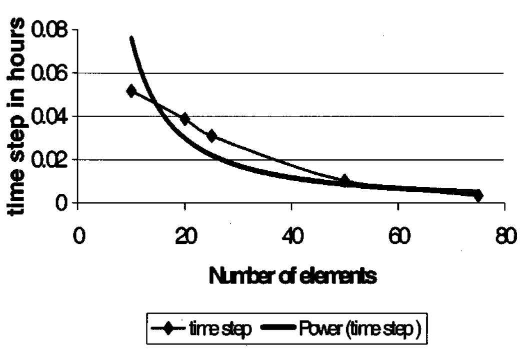

Fig. 10. Regression of time step versus number of elements for Case 2

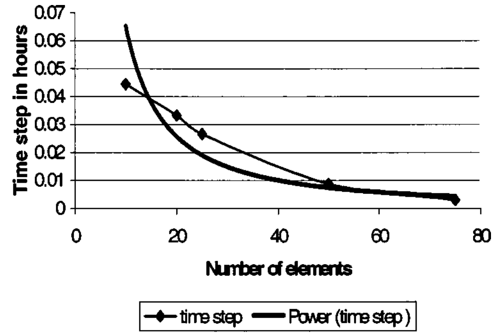

Fig. 11. Regression of time step versus number of elements for Case 3

$$\Delta t_1 = \frac{t_c}{N^{1.16}} \tag{13}$$

$$\Delta t_2 = \frac{0.81 \cdot t_c}{N^{1.08}} \tag{14}$$

$$\Delta t_3 = \frac{0.96 \cdot t_c}{N^{1.27}} \tag{15}$$

$$\Delta t_4 = \frac{1.66 \cdot t_c}{N^{1.43}} \tag{16}$$

where N = number of elements; and t_c = time of concentration in hours, computed as indicated in Eqs. (7) and (11).

Evaluation of Dynamic Time Step Criteria for Steady Rainfall

In order to test the new criteria, another numerical experiment was designed using 0.5, 1.0, 1.25, and 1.5 times the dynamic time step determined from Eqs. (13)–(16). The same mesh sizes used in the original experiment were used in the validation: 5, 10, 20, 25, 50, and 75 elements. The experiment was designed based on bench-scale laboratory experiments run by Yu and McNown (1964). The analytical solution matched the laboratory data perfectly (Agiralioglu and Singh 1981). The experiment consists of a slope of $L = 144$ m (472.5 ft); $S_0 = 0.02$; and roughness $n = 0.009$. The rainfall event simulated was $r_e = 19$ cm/h (1.731E-4 ft/s) applied for 0.133 hours. These values of r_e and α were used in the cases where r_e and α were constant. For the spatially varying r_e and α, the following equations were used:

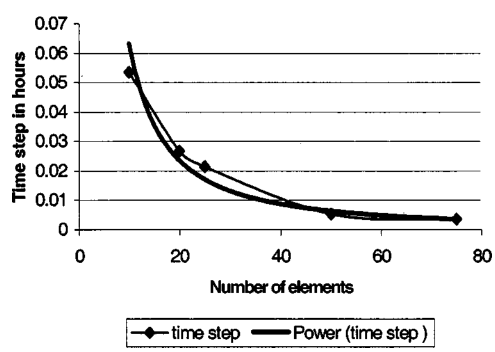

Fig. 12. Regression of time step versus number of elements for Case 4

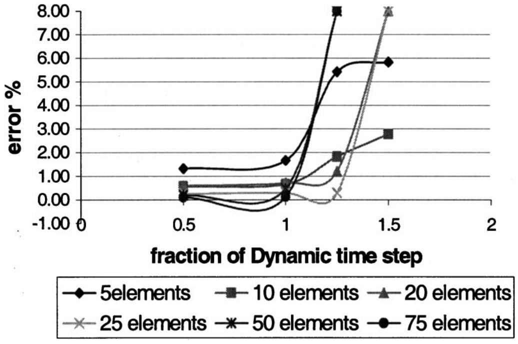

Fig. 13. Case 1 evaluation results

$$r_e = -0.2736 \cdot x + 39.51$$

$$\alpha = 8.0909 \cdot e^{0.0079x}$$

This α accounts for a variation in Manning's roughness coefficient n from a value of 0.02–0.04 and also allows for a variation in slope from 0.04 to 0.13. t_c was found to be 0.054 h for case 1 and 0.07 h for cases 2, 3, and 4. The time step was calculated using Eqs. (13)–(16) for each scenario as appropriate. The results showed that, when the dynamic time step was used, the error remained below 5%. The 1.25 and 1.5 multiples of the dynamic time step resulted in instability and the solution did not converge for some mesh sizes. Use of the 0.5 multiple of the dynamic time step produced errors that were insignificantly lower than the error for the time step itself, even though it needed larger input files and longer computational time. The results are shown in Figs. 13–16.

The development cases used Manning's roughness n values of 0.02–0.04, which corresponds to a surface slightly rougher than bare clay–loam soil (eroded) (Woolhiser 1975). The test case had a Manning's roughness value n of 0.009, which corresponds to a surface slightly smoother than concrete, asphalt, or bare sand (Woolhiser 1975). In order to test the dynamic time step criteria on rougher surfaces that would reflect vegetated conditions, the writers used a Manning's roughness n of 0.1, which corresponds to areas of sparse vegetation or short grass prairies. The dynamic time step proved to be adequate in this situation, while 1.5 times the dynamic time step resulted in instability (Fig. 17).

Evaluation of Dynamic Time Step Criteria for Unsteady Rainfall

The dynamic time step criteria were tested for unsteady rainfall conditions. The analytical solution for this type of storm is different from the steady rainfall situation. The problem simulated and the analytical solution used were taken from Parlange et al. (1981). This solution is only valid for case 1, where r_e is uniform in space. The storm used for this simulation is of the following form:

$$r_e = r_{e0} e^{(-t/r)} (1 - e^{(-t/\tau)}) \tag{17}$$

where r_{e0} and τ are constants. This storm is maximum at $t = t_m$, where $t_m/\tau = \mathrm{LN}(2)$. For this case, $r_{e0} = 0.762$ cm/h; and $\tau = 0.12$ hours.

The analytical solution of this problem is

$$h = (\tfrac{1}{2} - e^{-t/\tau} + \tfrac{1}{2} e^{-2t/\tau}) r_{e0} \tau \tag{18}$$

for

$$x \geqslant \beta \alpha r_{e0}^{2/3} \tau^{\beta} \int_0^{t/\tau} \left(\frac{1}{2} - e^{-\epsilon} + \frac{1}{2} e^{-2\epsilon} \right) d\epsilon \tag{19}$$

where ϵ is a dummy variable.

When inequality (19) is not satisfied, then

$$h = (e^{t_0/\tau} - e^{-t/\tau} - \tfrac{1}{2} e^{-2t_0/\tau} + \tfrac{1}{2} e^{-2t/\tau}) r_{e0} \tau \tag{20}$$

and

$$x = \beta a \alpha r_{e0}^{2/3} \tau^{\beta} \int_0^{t/\tau} \left(e^{t_0/\tau} - e^{-t/\tau} - \frac{1}{2} e^{-2t_0/\tau} + \frac{1}{2} e^{-2t/\tau} \right) d\epsilon \tag{21}$$

where an arbitrary value $t_0 < t$ is chosen, giving the value h for the calculated x.

The time of concentration is calculated by replacing x by L in Eq. (19) and calculating t. The solution required a simple numeri-

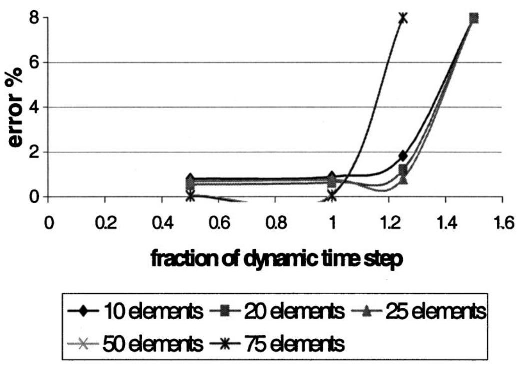

Fig. 14. Case 2 evaluation results

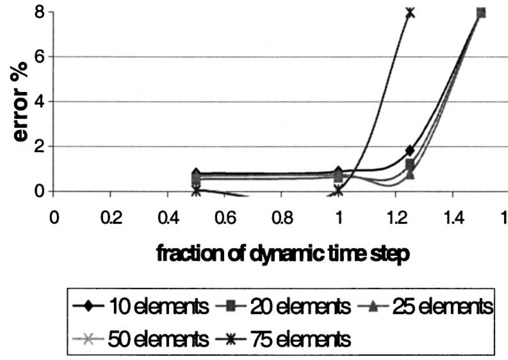

Fig. 15. Case 3 evaluation results

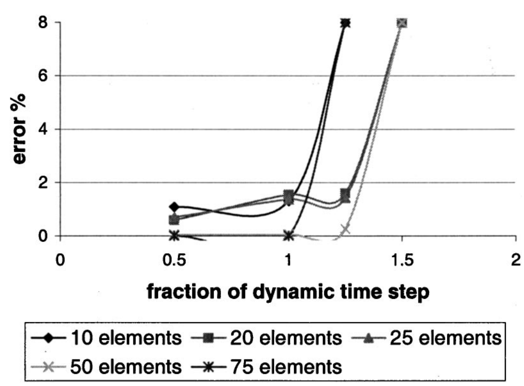

Fig. 16. Case 4 evaluation results

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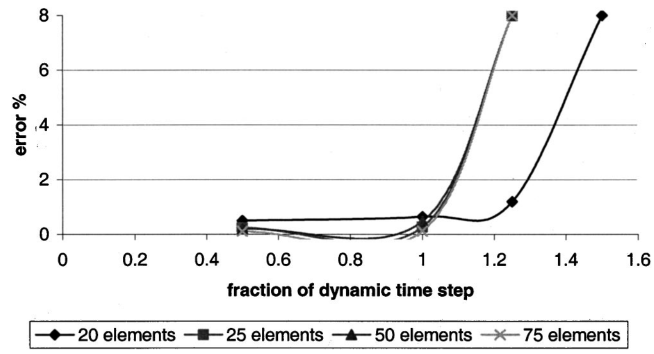

Fig. 17. Case with Manning's roughness $n=0.1$ evaluation results

cal integration scheme and Simpson's 1/3 Rule was used in this case. This solution was compared with the numerical simulation for the dynamic time step. The results in Fig. 18 show that the simulation using the dynamic time step had a very small error, while the use of 1.5 times the dynamic time step resulted in instability.

Implementation and Input-Free Environment

The dynamic time step equations developed in this study could be integrated within any hydrologic computer program, utilizing the kinematic wave theory. These time steps are functions of the grid size and the time of concentration, the latter being a function of rainfall, slope, and roughness parameters that are readily available. The time step then can be calculated automatically, depending on the mesh size that the user chooses. This will preclude the trial-and-error procedure, or the need for very experienced users that usually accompanies the selection of a time step to integrate the problem accurately.

Summary and Conclusions

Results of the kinematic wave overland flow solution using different time steps showed that the conventionally used stability criterion known as the Courant condition fails to give a time step estimate that ensures stable and accurate numerical solutions. Accordingly, new accuracy-based dynamic time step estimates for the overland flow kinematic wave solution were developed. The newly developed dynamic time step estimates are functions of grid size, watershed slope, roughness, and time of concentration.

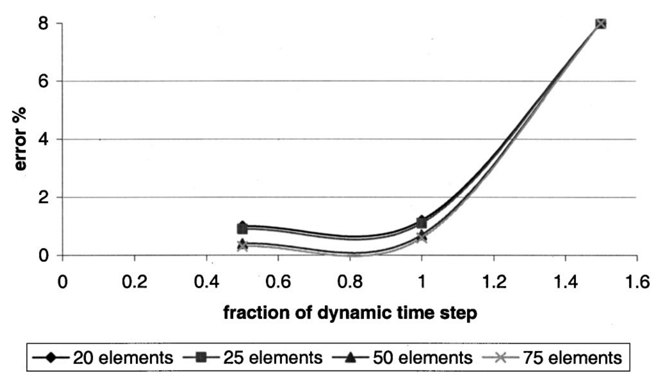

Fig. 18. Unsteady rain test case evaluation results

For each simulation, characterized by a problem boundary and initial conditions and mesh size, an optimal time step that integrates the problem within 5% error was determined. The series of mesh sizes and corresponding optimal time steps were used to develop the dynamic time step. The time step criteria were tested on a variety of problems, including steady and unsteady rain, and proved to be adequate for accurate and stable results within an efficient computational time. The criteria can be easily integrated in flow routing models to choose the optimal time step with minimal user input.

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Notation

The following symbols are used in this paper:

$[B]$ = gradient matrix;
$[C]$ = capacitance matrix;
$\{F\}$ = forcing term vector;
F = Froude number;
g = gravitational acceleration;
h = flow depth;
n = Manning's roughness parameter;
q = flow rate per unit area;
q_a = analytical solution for q;
q_n = numerical solution for q;
r_e = rainfall excess;
r_{e0}, τ = constants;
S_f = friction slope;
S_0 = element slope;
t_c = time of concentration;
t_m = time of maximum storm intensity;
V = flow velocity;
α = Manning's slope and roughness factor;
β = 5/3;
Δt = time step;
η, ξ = dummy variables; and
θ = finite difference scheme factor.

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